4. V. I. Krylov, V. V. Bobkov, and P. I. Monastyrskii, Computational Methods [in Russian], Vol. 2, Nauka, Moscow (1977).
5. V. P. Yastrebov, "Similarity solutions on the propagation of longitudinal waves in nonlinear media," Zh. Prikl. Mekh. Tekh. Fiz., No. 4 (1985).

TWO-DIMENSIONAL INVERSE PROBLEM OF NONLINEAR ELASTICITY THEORY FOR A HARMONIC MATERIAL

L. G. Dobordzhginidze UDC 539.3

For a material of harmonic type [1] we consider a two-dimensional inverse problem of nonlinear elasticity theory concerned with the determination of the contour of a hole having uniform strength. This problem was solved in [2] in the linear classical case.

1. Let us assume that the nonlinearly elastic medium under consideration here occupies the plane of the variable z = x + iy, weakened by a curvilinear hole. We assume also that constant normal stresses are applied to the contour L of this hole [3]:

$$\sigma_n = P_0,\ \tau_n = 0, \tag{1.1}$$

and that there is a biaxial tension along the coordinate axes at infinity:

$$\sigma_x^{(\infty)} = P_1,\ \sigma_y^{(\infty)} = P_2. \tag{1.2}$$

Subject to these conditions, we wish to find the shape and location of the contour L so that the tangential stress σ_t will be constant at all of its points:

$$\sigma_t = \sigma \tag{1.3}$$

(σ is constant but unknown).

To solve the problem we make use of complex representations for the stress and deformation fields in terms of functions $\varphi(z)$ and $\psi(z)$, analytic in the physical domain S under consideration (see [4, 5]):

$$\sigma_x + \sigma_y + 4\mu = \frac{\lambda + 2\mu}{\sqrt{J}} q\Omega(q), \tag{1.4}$$

$$\sigma_y - \sigma_x - 2i\tau_{xy} = -\frac{4(\lambda + 2\mu)}{\sqrt{J}} \frac{\Omega(q)}{q} \frac{\partial z^*}{\partial z} \frac{\partial z^*}{\partial \bar{z}};$$

$$\frac{\partial z^*}{\partial z} = \frac{\mu}{\lambda + 2\mu} \varphi'^2(z) + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\varphi'(z)}{\overline{\varphi'(z)}},\ \frac{\partial z^*}{\partial \bar{z}} = -\frac{\lambda + \mu}{\lambda + 2\mu} \left[\frac{\varphi(z)\overline{\varphi''(z)}}{\overline{\varphi'^2(z)}} - \overline{\psi'(z)} \right]; \tag{1.5}$$

$$\sqrt{J} = \frac{\partial z^*}{\partial z} \frac{\partial \bar{z}^*}{\partial \bar{z}} - \frac{\partial z^*}{\partial \bar{z}} \frac{\partial \bar{z}^*}{\partial z},\ q = 2\left|\frac{\partial z^*}{\partial z}\right|,\ \Omega(q) = q - \frac{2(\lambda + \mu)}{\lambda + 2\mu} \tag{1.6}$$

(λ, μ are the Lamé elastic constants). For large $|z|$ these functions have the asymptotics

$$\varphi(z) = a_0 z + O(z^{-1}),\ \psi(z) = b_0 z + O(z^{-1}) \tag{1.7}$$

(a_0, and b_0 are known constants [6]);

$$a_0 = \left[\frac{\lambda + \mu}{\mu} \frac{2\mu(P_1 + P_2) + P_1 P_2 + 4\mu^2}{\lambda(P_1 + P_2) - P_1 P_2 + 4\mu(\lambda + \mu)} \right]^{1/2}, \tag{1.8}$$

$$b_0 = \frac{(\lambda + 2\mu)(P_1 - P_2)}{\lambda(P_1 + P_2) - P_1 P_2 + 4\mu(\lambda + \mu)}.$$

Comparing the relations (1.4), we obtain the equation

$$\frac{\partial \bar{z}^*}{\partial z} = \frac{\sigma_x - \sigma_y - 2i\tau_{xy}}{\sigma_x + \sigma_y + 4\mu} \frac{\partial z^*}{\partial z}, \tag{1.9}$$

using this, we have, based on relations (1.4)-(1.6), after some calculations,

Tbilisi. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 155-159, January-February, 1988. Original article submitted October 20, 1986.

$$\sigma_x + \sigma_y + 4\mu = \frac{4\mu(\lambda+2\mu)\,|\varphi'^2(z)|}{(1-A)\,[\mu\,|\varphi'^2(z)|+\lambda+\mu]}; \tag{1.10}$$

$$A = \frac{(\sigma_y-\sigma_x)^2+4\tau_{xy}^2}{(\sigma_x+\sigma_y+4\mu)^2}. \tag{1.11}$$

2. We map the physical domain S under consideration conformally and biuniquely by means of the function

$$z = \omega(\zeta)\ (\omega(\zeta) = R\zeta + O(\zeta^{-1})\ \text{for large}\ |\zeta|) \tag{2.1}$$

onto the exterior of the unit disk $|\zeta| > 1$ in the plane of the auxiliary variable $\zeta = \xi + i\eta$; we retain the previous notation for the quantities considered in the transformed domain. Further, we shall use the relations [3]

$$\breve{\rho\rho} + \breve{\theta\theta} = \sigma_x + \sigma_y,\ \breve{\theta\theta} - \breve{\rho\rho} + 2i\breve{\rho\theta} = (\sigma_y - \sigma_x + 2i\tau_{xy})e^{2i\alpha}. \tag{2.2}$$

Then from relations (1.1), (1.3), (1.10), (1.11), (2.1), and (2.2) we obtain the following, valid on the unit circle $|\zeta| = 1$, which we denote by γ:

$$\left|\frac{\varphi'^2(\sigma)}{\omega'^2(\sigma)}\right| = \frac{(\lambda+\mu)(P_0+2\mu)(\sigma+2\mu)}{\mu\,[(\lambda+2\mu)(P_0+\sigma+4\mu)-(P_0+2\mu)(\sigma+2\mu)]} = h_0^2\ \text{on}\ \gamma. \tag{2.3}$$

From the boundary condition (2.3) we obtain, upon taking relations (1.7) and (2.1) into account, after obvious arguments,

$$\varphi'(\zeta) = h_0\omega'(\zeta) \tag{2.4}$$

everywhere in the domain $|\zeta| > 1$.

Comparing equation (2.4) with the first of the relations (1.7), we have

$$a_0 = h_0, \tag{2.5}$$

which, based on equations (1.8), yields a relationship between the stresses in question in the form

$$\sigma = \frac{4\mu^2(P_1+P_2)+(P_0+4\mu)P_1P_2-4\mu^2P_0}{P_0(P_1+P_2)-P_1P_2+4\mu(P_0+\mu)}. \tag{2.6}$$

We return now to the last relation (1.4) and in it take into account relations (1.5), (2.2), and (2.4). Then

$$\psi'(\sigma) = A_0\bar{\sigma}^2\overline{\omega'(\sigma)}\ \text{on}\ \gamma, \tag{2.7}$$

where A_0 is a constant given by the expression

$$A_0 = \frac{(\lambda+2\mu)(P_1+P_2+4\mu)^2(P_0-\sigma)(\sigma+2\mu)(P_0+2\mu)}{(\sigma+P_0+4\mu)^2\,[\lambda(P_1+P_2)-P_1P_2+4\mu(\lambda+\mu)]\,[2\mu(P_1+P_2)+P_1P_2+4\mu^2]}. \tag{2.8}$$

Next, we introduce the notation:

$$F^-(\zeta) = \psi'(\zeta),\ F^+(\zeta) = \frac{A_0}{\zeta^2}\bar{\omega}'\left(\frac{1}{\zeta}\right). \tag{2.9}$$

Here $F^+(\zeta)$ is a function holomorphic in the disk $|\zeta| < 1$, except for the point $\zeta = 0$, where it has a second-order pole, while $F^-(\zeta)$ is holomorphic in the domain $|\zeta| > 1$ and is a bounded function at $\zeta = \infty$. Then

$$F^+(\sigma) = F^-(\sigma)\ \text{on}\ \gamma. \tag{2.10}$$

We obtain from this, using a well-known theorem of Liouville [3] and the relations (1.7) and (2.1), a single function $F(\zeta)$, holomorphic in the extended plane $\zeta = \xi + i\eta$ (with the exception of the point $\zeta = 0$), in the form

$$F(\zeta) = b_0R + A_0R/\zeta^2. \tag{2.11}$$

Now, on the basis of relations (2.10) and (2.11), we obtain a solution of problem (2.7):

$$\omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right),\ \frac{\psi'(\zeta)}{\omega'(\zeta)} = A_0\frac{1-m\zeta^2}{\zeta^2-m}; \tag{2.12}$$

$$m = -b_0/A_0. \tag{2.13}$$

On the right side of equation (2.13) we insert the values of σ, given by equation (2.6):

TABLE 1

P_2/μ	m_2/m_1				
	P_1/μ				
	0	0,2	0,4	0,6	0,8
	1	0	1,15	1,17	1,19
0,2	1	0	0,88	0,87	0,86
	1	1,15	0	1,31	1,35
0,4	1	0,88	0	0,81	0,75
	1	1,17	1,31	0	1,51
0,6	1	0,87	0,81	0	0,74

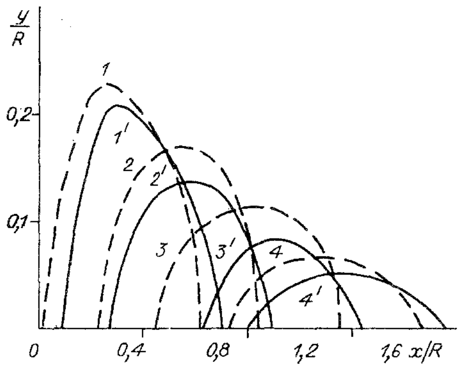

Fig. 1

$$m=\frac{(P_1-P_2)(2\mu+P_0)}{2\mu(P_1+P_2-2P_0)-(P_1+P_2)P_0+2P_1P_2}. \quad (2.14)$$

Consequently, the desired hole contours of equal strength constitute a set of similar ellipses with eccentricity m given by equation (2.14).

According to the linear classical theory, we have the relation

$$m=(P_1-P_2)/(P_1+P_2-2P_0) \quad (2.15)$$

for the eccentricity and $\sigma = P_1 + P_2 - P_0$ for the constant σ_t on L.

Table 1 gives values of the ratio m_2/m_1 for various values of P_i/μ, (i = 1, 2), at first in the biaxial tension field and then in the biaxial compressive field; m_1 and m_2 are the eccentricities corresponding to the linear (relation (2.15)) and nonlinear (relation (2.14)) theories, respectively.

As is evident from Table 1, in the case of a biaxial tension the eccentricity of the desired ellipses of equal strength increases, while in the case of compressive stresses a decrease in this geometric characteristic is observed when compared with the linear classical case. Hence, in the latter case, the form of the ellipse in the nonlinear theory is closer to a circle than in the linear classical theory. This fact is of great practical significance.

After determining $\omega(\zeta)$, $\varphi'(\zeta)$, and $\psi'(\zeta)$, we obtain the functions $\varphi(\zeta)$, $\psi(\zeta)$ from relations (2.4) and (2.12) in the form $\varphi(\zeta) = a_0R(\zeta + m/\zeta)$, $\psi(\zeta) = -A_0R(m\zeta + 1/\zeta)$. Solution of our problem is now complete.

3. We now consider the problem for two holes. In this case we assume that the function $z = \omega(\zeta)$ effects a conformal and biunique mapping of the given multiply connected domain onto the exterior of cuts $\Gamma = \Gamma_1 + \Gamma_2$ ($\Gamma_1 = a_1b_1$, $\Gamma_2 = a_2b_2$), located on the real $O\xi$-axis of the plane of the variable $\zeta = \xi + i\eta$.

Then, following reasoning similar to that in Sec. 2, we can see that the solution of the stated problem has the form ($a_1=-b$, $b_1=-a$, $a_2=a$, $b_2=b$)

$$\varphi(\zeta)=a_0\omega(\zeta),\ \omega(\zeta)=\frac{R(A_0-b_0)}{2A_0}\zeta+\frac{R(A_0+b_0)}{2A_0}\left[bE(\varphi,k)-\left(b+\frac{c}{b}\right)F(\varphi,k)\right],$$

$$\psi(\zeta)=\frac{R(b_0-A_0)}{2}\zeta+\frac{R(A_0+b_0)}{2}\left[bE(\varphi,k)-\left(b+\frac{c}{b}\right)F(\varphi,k)\right].$$

Here R and c are arbitrary real constants; F and E are elliptic integrals of the first and second kind, respectively:

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi, \quad \varphi = \arcsin \frac{\zeta}{a}, \quad k = \frac{a}{b}.$$

Next, for a comparison with the linear classical case we put, without loss of generality, $a_1 = -2$, $b_1 = -1$, $a_2 = 1$, $b_2 = 2$. Then, following the reasoning used in [2], we determine the constant c, and then also the form of the contours in the form

$$c = -\frac{4E(\sqrt{3}/2)}{F(\sqrt{3}/2)} = -2.246, \quad x = -R\left(0.23 + 1.23\frac{b_0}{A_0}\right) + \frac{R}{2}\left(1 - \frac{b_0}{A_0}\right)(\xi - 1), \quad 1 < \xi < 2, \quad y = -\frac{R}{2}\left(1 + \frac{b_0}{A_0}\right)\left[2E\left(\varphi_0, \frac{\sqrt{3}}{2}\right) - 1.123\, F\left(\varphi_0, \frac{\sqrt{3}}{2}\right) - \frac{1}{\xi}\sqrt{(4 - \xi^2)(\xi^2 - 1)}\right], \quad \varphi_0 = \arcsin \frac{2\sqrt{\xi^2 - 1}}{\sqrt{3}\,\xi},$$

$$m = \frac{b_0}{A_0} = \frac{(P_1 - P_2)(\sigma + P_0 + 4\mu)^2 (P_1 + 2\mu)(P_2 + 2\mu)}{(P_0 - \sigma)(P_1 + P_2 + 4\mu)^2 (\sigma + 2\mu)(P_0 + 2\mu)}.$$

According to the linear classical theory, $m = (P_2 - P_1)/(\sigma - P_0)$. Figure 1 shows the graphs for a family of contours of holes of equal strength for various values of $m = b_0/A_0$ ($x > 0$, $y > 0$). Curves 1 to 4 correspond to the linear case with $m_1 = -0.2$; -0.4; -0.6; and -0.8, while curves 1' to 4' are for the nonlinear case with $m_2 = -0.26$; -0.50; -0.61; and -0.88.

Analysis of these curves shows that the contours for holes of equal strength, constructed in accordance with the nonlinear and linear theories, differ. In some cases involving elastic equilibrium this difference may prove to be significant.

LITERATURE CITED

1. F. John, "Plane strain problems for a perfectly elastic material of harmonic type," Commun. Pure Appl. Math., 13, No. 2 (1960).
2. G. P. Cherepanov, "Inverse problems of the planar theory of elasticity," Prikl. Mat. Mekh., 38, No. 6 (1974).
3. N. I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity, Noordhoff, Groningen (1953).
4. A. I. Lur'e, Nonlinear Theory of Elasticity [in Russian], Nauka, Moscow (1980).
5. L. G. Dobordzhginidze, "A two-dimensional nonlinear problem of distribution of stresses around holes," Prikl. Mekh., 13, No. 9, (1982).
6. L. G. Dobordzhginidze, "Complex representation of displacements and stresses for a nonlinearly elastic material of harmonic type," Tr. Tbilis. Mat. Inst., 61, (1979).